

Shell Crossing Singularities in Quasi-Spherical Szekeres Models

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We investigate the occurrence of shell crossing singularities in quasi-spherical Szekeres dust models with or without a cosmological constant. We study the conditions for shell crossing singularity both from physical and geometrical point of view and they are in agreement.

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I. INTRODUCTION

In the study of gravitational collapse, we always encounter with two types of singularities – shell focusing singularity and shell crossing singularity. In Tolman-Bondi-Lemaître (TBL) dust model, these two kinds of singularities will corresponds to $R = 0$ and $R' = 0$ respectively. A shell focusing singularity (i.e., $R = 0$) on a shell of dust occurs when it collapses at or starts expanding from the centre of matter distribution. The instant at which a shell at the radial co-ordinate r will reach the centre of the matter distribution should be a function of r . So different shells of dust arrive at the centre at different times and there is always a possibility that any two shells of dust cross each other at a finite radius in course of their collapse. In this situation the comoving system breaks down, both the matter density and kretschmann scalar diverge [1,2] and one encounters the shell crossing singularity. If one treats it as the boundary surface then the region beyond it is unacceptable since it has negative density. It is therefore, of interest to find conditions which guarantee that no shell crossing will occur.

In TBL model, shell crossing singularity has been studied by several authors [1-9]. Also Goncalves [7] studied the occurrence of shell crossing in spherical weakly charged dust collapse in the presence of a non-vanishing cosmological constant. The positive cosmological constant model conceivably prevent the occurrence of shell crossing thereby allowing at least in principle for a singularity free ‘bounce’ model. Nolan [8] derive global weak solutions of Einstein’s equations for spherically symmetric dust-filled space-times which admit shell crossing singularities. Recently, Hellaby et al [9] investigate the anisotropic generalization of the wormhole topology in the Szekeres model. In this work, we have studied the shell crossing singularity in Szekeres model of the space-time both from physical and geometrical point of view. In section II, we derive the basic equations in shell focusing and shell crossing singularities. We study the physical conditions and geometrical features of shell crossing singularities in sections III and IV respectively. Finally the paper ends with a short discussion in section V.

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II. BASIC EQUATIONS IN SHELL FOCUSING AND SHELL CROSSING SINGULARITIES

Recently, we have presented dust solutions for $(n + 2)$ -dimensional Szekeres' space-time model with metric ansatz [10]

$$ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} \sum_{i=1}^n dx_i^2 \quad (1)$$

where α and β are functions of all the $(n + 2)$ space-time co-ordinates. If we assume that $\beta' (= \frac{\partial \beta}{\partial r}) \neq 0$, then the explicit form of the metric coefficients are

$$e^\beta = R(t, r) e^{\nu(r, x_1, \dots, x_n)} \quad (2)$$

and

$$e^\alpha = \frac{R' + R \nu'}{\sqrt{1 + f(r)}} \quad (3)$$

where

$$e^{-\nu} = A(r) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n B_i(r) x_i + C(r) \quad (4)$$

and R satisfied the differential equation

$$\dot{R}^2 = f(r) + \frac{F(r)}{R^{n-1}} + \frac{2\Lambda}{n(n+1)} R^2. \quad (5)$$

Here Λ is the cosmological constant, $f(r)$ and $F(r)$ are arbitrary functions of r alone; and the other arbitrary functions, namely $A(r)$, $B_i(r)$'s and $C(r)$ in equation (4) are algebraically related by the relation

$$\sum_{i=1}^n B_i^2 - 4AC = -1. \quad (6)$$

It is to be noted that the r -dependence of these arbitrary functions A , B_i and C play an important role in characterizing the geometry of the $(n + 1)$ -dimensional space. In fact, the choice $A(r) = C(r) = \frac{1}{2}$ and $B_i(r) = 0$ ($\forall i = 1, 2, \dots, n$) reduce the space-time metric (1) to the usual spherically symmetric TBL form

$$ds^2 = dt^2 - \frac{R'^2}{1 + f(r)} dr^2 - R^2 d\Omega_n^2. \quad (7)$$

by the co-ordinate transformation

$$\begin{aligned} x_1 &= \sin \theta_n \sin \theta_{n-1} \dots \dots \sin \theta_2 \cot \frac{1}{2} \theta_1 \\ x_2 &= \cos \theta_n \sin \theta_{n-1} \dots \dots \sin \theta_2 \cot \frac{1}{2} \theta_1 \\ x_3 &= \cos \theta_{n-1} \sin \theta_{n-2} \dots \dots \sin \theta_2 \cot \frac{1}{2} \theta_1 \\ &\dots \dots \dots \dots \dots \dots \dots \dots \\ x_{n-1} &= \cos \theta_3 \sin \theta_2 \cot \frac{1}{2} \theta_1 \\ x_n &= \cos \theta_2 \cot \frac{1}{2} \theta_1 \end{aligned}$$

Hence in the subsequent discussion we shall restrict ourselves to the quasi-spherical space-time which is characterized by the r dependence of the function ν (i.e., $\nu' \neq 0$). The expression for energy density due to dust matter using the Einstein equations is

$$\rho(t, r, x_1, \dots, x_n) = \frac{n}{2} \frac{F' + (n+1)F\nu'}{R^n(R' + R\nu')}. \quad (8)$$

The space-time singularity will occur when either (i) $R = 0$ i.e., $\beta = -\infty$ or (ii) $R' + R\nu' = 0$ i.e., $\alpha = -\infty$. The standard terminology for spherical collapse suggests that the first case corresponds to shell focusing singularity while in the second case we have a shell-crossing singularity. In the following we shall discuss the situations for shell crossing singularity.

Suppose the collapse develops at the initial hypersurface $t = t_i$ where we assume $R(t_i, r)$ to be a monotonically increasing function of r . So, without any loss of generality, we can label the dust shells by the choice $R(t_i, r) = r$. Hence the expression for the initial density distribution is given by

$$\rho_i(r, x_1, \dots, x_n) = \rho(t_i, r, x_1, \dots, x_n) = \frac{n}{2} \frac{F' + (n+1)F\nu'}{r^n(1 + r\nu')} \quad (9)$$

If we consider that the collapsing process starts from a regular initial hypersurface then the function ρ_i must be non-singular (and also positive from physical point of view). Moreover the local flatness property of the space-time near $r = 0$ demands $f(r) \rightarrow 0$ as $r \rightarrow 0$. Then in order to \dot{R}^2 to be bounded as $r \rightarrow 0$ we must have $F(r) \sim O(r^m)$ where $m \geq n-1$ (see eq. (5)). On the other hand, for small r , $\rho_i(r) \simeq \frac{n}{2} \frac{F' + (n+1)F\nu'}{r^n}$ and consequently, for regular $\rho_i(r)$ near $r = 0$, we must have $F(r) \sim O(r^{n+1})$ and $\nu' \sim O(\frac{1}{r})$. Hence, starting from a regular initial hypersurface, we can express $F(r)$ and $\rho_i(r)$ as a power series near $r = 0$ as [11]

$$F(r) = \sum_{j=0}^{\infty} F_j r^{n+j+1} \quad (10)$$

and

$$\rho_i(r) = \sum_{j=0}^{\infty} \rho_j r^j. \quad (11)$$

As ν' appears in the expression for the density as well as in the metric coefficient, so we can write [11]

$$\nu'(r) = \sum_{j=-1}^{\infty} \nu_j r^j \quad (12)$$

where $\nu_{-1} > -1$.

Now, using these series expansions in equation (9) we have the following relations between the coefficients,

$$\rho_0 = \frac{n(n+1)}{2} F_0, \quad \rho_1 = \frac{n}{2} \left(n+1 + \frac{1}{1+\nu_{-1}} \right) F_1,$$

$$\rho_2 = \frac{n}{2} \left[\left(n + 1 + \frac{2}{1 + \nu_{-1}} \right) F_2 - \frac{F_1 \nu_0}{(1 + \nu_{-1})^2} \right],$$

$$\rho_3 = \frac{n}{2} \left[\left(n + 1 + \frac{3}{1 + \nu_{-1}} \right) F_3 - \frac{2F_2 \nu_0}{(1 + \nu_{-1})^2} - \frac{(1 + \nu_{-1})\nu_1 - \nu_0^2}{(1 + \nu_{-1})^3} F_1 \right]$$

and so on.

Now in order to form a singularity from the gravitational collapse of dust, all portions of the dust cloud should collapse i.e., $\dot{R} \leq 0$. Let us denote by $t_{sf}(r)$ and $t_{sc}(r)$ as the time for shell-focusing and shell-crossing singularities occurring at radial coordinate r . Hence we have the relations

$$R(t_{sf}, r) = 0 \quad (13)$$

and

$$R'(t_{sc}, r) + R(t_{sc}, r)\nu'(r, x_1, x_2, \dots, x_n) = 0. \quad (14)$$

Note that ' t_{sc} ' may also depend on x_1, x_2, \dots, x_n .

III. PHYSICAL CONDITIONS FOR SHELL CROSSING SINGULARITY

We shall now make a comparative study of shell focusing and shell crossing singularity time and find conditions in favour (or against) of formation of shell crossing singularity for the following different choices:

(i) $f(r) = 0, \Lambda = 0$:

In this case equation (5) can be integrated to give

$$R^{\frac{n+1}{2}} = r^{\frac{n+1}{2}} - \frac{n+1}{2} \sqrt{F(r)} (t - t_i) \quad (15)$$

So $R(t_{sf}, r) = 0$ results

$$t_{sf}(r) = t_i + \frac{2}{(n+1)\sqrt{F(r)}} r^{\frac{n+1}{2}} \quad (16)$$

Now to avoid the shell crossing singularity either all shells will collapse at the same time (i.e., t_{sf} is independent of r) or larger shell will collapse at late time (i.e., $t_{sf}(r)$ is a monotone increasing function of r). These two conditions can be combined as

$$t'_{sf}(r) \geq 0$$

or equivalently from equation (16)

$$\frac{F'(r)}{F(r)} \leq \frac{n+1}{r} \quad (17)$$

Now combining equations (14) and (15) we have

$$t_{sc}(r) - t_{sf}(r) = \frac{2r^{\frac{n+1}{2}} \left\{ \frac{n+1}{r} - \frac{F'(r)}{F(r)} \right\}}{(n+1)\sqrt{F(r)} \left\{ \frac{F'(r)}{F(r)} + (n+1)\nu' \right\}} \quad (18)$$

But if it is so happen that $R' + R\nu' = 0$ is a regular extremum for β , then we must have finite ρ . This implies from equation (8) that $F' + (n+1)F\nu' = 0$. Hence, if there is no shell crossing singularity corresponding to equation (14) we must have two possibilities:

$$\text{either (a) } \frac{F'(r)}{F(r)} + (n+1)\nu' = 0 \quad \text{and} \quad \frac{n+1}{r} - \frac{F'(r)}{F(r)} = 0 \quad (19)$$

$$\text{or (b) } \frac{F'(r)}{F(r)} + (n+1)\nu' = 0 \quad \text{and} \quad \frac{n+1}{r} - \frac{F'(r)}{F(r)} > 0 \quad (20)$$

For the first choice t_{sf} is constant, so all shells collapse simultaneously while for the second choice t_{sf} is a monotonic increasing function of r and there is an infinite time difference between the occurrence of both type of singularities.

The value of R at $t = t_{sc}(r)$ is

$$\{R(t_{sc}, r)\}^{\frac{n+1}{2}} = \frac{r^{\frac{n+1}{2}} \left\{ \frac{F'(r)}{F(r)} - \frac{n+1}{r} \right\}}{\left\{ \frac{F'(r)}{F(r)} + (n+1)\nu' \right\}} \quad (21)$$

Therefore as an complementary event, the conditions for occurrence of shell crossing singularity are $R' + R\nu' = 0$, $\rho = \infty$, $\dot{R} < 0$, $R > 0$, $t'_{sf} < 0$.

As $t'_{sf} < 0$ implies

$$\frac{F'(r)}{F(r)} > \frac{n+1}{r} \quad (22)$$

so $R > 0$ demands

$$\frac{F'(r)}{F(r)} + (n+1)\nu' > 0 \quad (23)$$

Hence we have

$$F(r) \sim r^l \quad \text{and} \quad e^\nu \sim r^p \quad (24)$$

for shell crossing singularity with $l > (n+1)$ and $(n+1)p > -l$.

(ii) $f(r) = 0$, $\Lambda \neq 0$:

This choice will give the solution to equation (5) as [11]

$$t = t_i + \sqrt{\frac{2n}{(n+1)\Lambda}} \left[\text{Sinh}^{-1} \left(\sqrt{\frac{2\Lambda r^{n+1}}{n(n+1)F(r)}} \right) - \text{Sinh}^{-1} \left(\sqrt{\frac{2\Lambda R^{n+1}}{n(n+1)F(r)}} \right) \right] \quad (25)$$

At the shell focusing time $t_{sf}(r)$, $R = 0$, hence we have

$$t_{sf}(r) = t_i + \sqrt{\frac{2n}{(n+1)\Lambda}} \sinh^{-1} \left(\sqrt{\frac{2\Lambda r^{n+1}}{n(n+1)F(r)}} \right) \quad (26)$$

Since to avoid the shell crossing singularity, $t_{sf}(r)$ should be an increasing (or a constant) function of time i.e., $t'_{sf}(r) \geq 0$, which implies the same condition (17) as in case (i) and is independent of Λ (whether zero or not). Further, substitution of equation (25) in equation (14) will give

$$t_{sc}(r) - t_{sf}(r) = \tanh^{-1} \left[\frac{\sqrt{\frac{2(n+1)\Lambda}{n}} t'_{sf}(r)}{\left\{ \frac{F'(r)}{F(r)} + (n+1)\nu' \right\}} \right] \quad (27)$$

But if there is no shell crossing singularity corresponding to $R' + R\nu' = 0$ (then it will correspond to an extremum of β) then ρ must be finite. This will be possible only when $F' + (n+1)F\nu' = 0$. But from eq.(27) it is permissible only when $t'_{sf} = 0$ i.e., t_{sf} is independent of r .

(iii) $f(r) \neq 0, \Lambda = 0, \dot{R}(t_i, r) = 0$ (*time symmetry*):

In this case explicit solution is possible only for five dimension (i.e., for $n = 3$) and the result as

$$R^2 = r^2 - \frac{F(r)}{r^2} (t - t_i)^2, \quad (28)$$

But the shell focusing condition $R(t_{sf}, r) = 0$ gives

$$t_{sf} = t_i + \frac{r^2}{\sqrt{F(r)}}.$$

So $t'_{sf} \geq 0$ will give

$$\frac{F'}{F} \leq \frac{4}{r} \quad (29)$$

Here the time difference between the two types of singularities is

$$t_{sc} - t_{sf} = \frac{r^2}{\sqrt{F}} \left[\sqrt{\frac{\nu' + \frac{1}{r}}{\frac{F'}{2F} + \nu' - \frac{1}{r}}} - 1 \right] \quad (30)$$

The r.h.s. of equation (30) always positive by the inequality (29).

IV. GEOMETRICAL FEATURES OF SHELL CROSSING SINGULARITY

Now we shall discuss the shell crossing singularity from geometrical point of view. We note that $R' + R\nu'$ (related to e^α) is always ≥ 0 . The equality sign corresponds to the shell crossing singularity. In fact geometrically, a shell crossing singularity (if it exists) is the locus of zeros of the function $R' + R\nu'$ (i.e., $\alpha = -\infty$). Now writing explicitly the function $R' + R\nu'$ using the solution (4) for ν we have

$$R' + R\nu' = e^\nu \left[(R'A - RA') \sum_{i=1}^n x_i^2 + \sum_{i=1}^n (R'B_i - RB'_i) x_i + (CR' - C'R) \right] \quad (31)$$

We see that (a detailed analysis is given in the **Appendix**) there will be no shell crossing singularity i.e., $R' + R\nu'$ will be positive definite if

$$\frac{R'^2}{R^2} > \sum_{i=1}^n B_i'^2 - 4A'C' = \psi(r) \text{ (say)} \quad (32)$$

Note that $R' + R\nu'$ will also be positive for $\frac{R'^2}{R^2} = \psi(r)$, provided $x_i \neq \frac{RB_i' - R'B_i}{2(R'A - RA')} = x_{0i}$, $i = 1, 2, \dots, n$. Thus shell crossing is a single point $(x_{01}, x_{02}, \dots, x_{0n})$ in the constant (t, r) -hypersurface (n dimensional). In other words, it is a curve in the t -constant $(n+1)$ -D hypersurface and a 2 surface in $(n+2)$ -D space-time.

When $\frac{R'^2}{R^2} < \psi(r)$ then shell crossing singularity lies on n -hypersphere in the n -dimensional x_i 's plane. This hypersphere has centre $(x_{01}, x_{02}, \dots, x_{0n})$ and radius

$$r_c = \frac{\sqrt{R^2 (\sum_{i=1}^n B_i'^2 - 4A'C') - R'^2}}{2(R'A - RA')} \quad (33)$$

In the above we have assumed $a = (R'A - RA')$ to be positive. However, if $a < 0$ and $\frac{R'^2}{R^2} < \psi(r)$ then also shell crossing singularity is possible and it lies on an n -hypersphere having same centre $(x_{01}, x_{02}, \dots, x_{0n})$ and radius r_c . Further, there will be no shell crossing singularity if the variables x_i 's lie inside the above n -hypersphere.

The above hypersphere is different from the hypersphere with $\nu' = 0$ i.e.,

$$A'(r) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n B_i'(r)x_i + C'(r) = 0 \quad (34)$$

So the shell crossing set intersects with the surface of constant r and t along the line (curve) $\frac{R'}{R} = -\nu' = \text{constant}$.

Now for positive density we note that $F' + (n+1)F\nu'$ and $R' + R\nu'$ must have the same sign. We now consider the case where both are positive (when both are negative, we just reverse the inequalities). When both are zero then it can happen for a particular value of x_i 's ($i = 1, 2, \dots, n$) if $\frac{F'}{(n+1)F} = \frac{R'}{R} = -\nu'$, which can not hold for all time. This is possible for all x_i if $F' = R' = \nu' = 0$. This implies that at some r , $F' = f' = A' = C' = B_i' = 0$ ($i = 1, 2, \dots, n$). Hence we choose

$$\frac{F'}{(n+1)F} > -\nu' \quad \text{and} \quad \frac{R'}{R} > -\nu' \quad (35)$$

Also from the solution (4) we have

$$-\nu' = \frac{A'(r) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n B_i'(r)x_i + C'(r)}{A(r) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n B_i(r)x_i + C(r)}$$

Now writing in a quadratic equation in x_1 we have for real x_1 ,

$$\nu'^2 + \left\{ 2(A' + A\nu') \sum_{k=2}^n x_k + \sum_{k=2}^n (B_k' + B_k\nu') \right\}^2 \leq \sum_{i=1}^n B_i'^2 - 4A'C'$$

So

$$\nu'^2|_{max} = \sum_{i=1}^n B_i'^2 - 4A'C'$$

Hence from (35) we have

$$\frac{F'}{(n+1)F} \geq \sqrt{\sum_{i=1}^n B_i'^2 - 4A'C'} , \quad \forall r \quad (36)$$

which implies $F' \geq 0, \forall r$. Now for $R' + R\nu' > 0$, we shall study the three possible choices separately.

(i) $f(r) = 0, \Lambda = 0$:

Here the solution for R can be written as

$$R^{\frac{n+1}{2}} = \frac{(n+1)}{2} \sqrt{F(r)} (t - a(r))$$

So as $t \rightarrow a$, $R^{\frac{n-1}{2}} R' + R^{\frac{n+1}{2}} \nu' \rightarrow -\sqrt{F(r)} a'(r)$ and as $t \rightarrow \infty$, $\frac{R'}{R} + \nu' \rightarrow \frac{F'}{(n+1)F} + \nu'$. Hence for $R' + R\nu' > 0$ we must have $a' < 0$ and $\frac{F'}{(n+1)F} > \sqrt{\sum_{i=1}^n B_i'^2 - 4A'C'}$.

(ii) $f(r) = 0, \Lambda = 0$:

The solution for R can be written as

$$R^{\frac{n+1}{2}} = \sqrt{\frac{n(n+1)F(r)}{2\Lambda}} \sinh \left[\sqrt{\frac{(n+1)\Lambda}{2n}} (t - a(r)) \right]$$

In this case as $t \rightarrow a$, $R^{\frac{n-1}{2}} R' + R^{\frac{n+1}{2}} \nu' \rightarrow -\sqrt{F(r)} a'(r)$ and as $t \rightarrow \infty$, $\frac{R'}{R} + \nu' \rightarrow \frac{F'}{(n+1)F} - \sqrt{\frac{2\Lambda F}{n(n+1)}} a'(r) + \nu'$. Thus for $R' + R\nu' > 0$ we must have $a'(r) < 0$ and $\frac{F'}{(n+1)F} - \sqrt{\frac{2\Lambda F}{n(n+1)}} a'(r) > \sqrt{\sum_{i=1}^n B_i'^2 - 4A'C'}$.

(iii) $f(r) \neq 0, \Lambda = 0, \dot{R}(t_i, r) = 0, n = 3$:

Here the solution for R is

$$R^2 = r^2 - \frac{F(r)}{r^2} (t - t_i)^2.$$

The limiting value of $\frac{R'}{R} + \nu'$ as $t \rightarrow \infty$ will be $\frac{F'}{2F} + \nu' - \frac{1}{r}$. Hence for $R' + R\nu' > 0$ we should have $\frac{F'}{2F} + \frac{1}{r} > \sqrt{\sum_{i=1}^n B_i'^2 - 4A'C'}$.

V. DISCUSSION

In the last two sections a details study of shell crossing singularity has been done for dust model with or without cosmological constant for Szekeres model of $(n+2)$ -D space-time. The physical conditions for shell crossing singularity are presented in section III. These conditions however do not depend on Λ (whether zero or not) and the form of the conditions are identical for the three cases presented there. For geometrical conditions the locus of shell crossing depends on the discriminant of the co-ordinate variables x_i 's ($i = 1, 2, \dots, n$). If both $\frac{R'}{R}$ and $\frac{F'}{(n+1)F}$ are greater than $\sqrt{\sum B_i'^2 - 4A'C'}$ then there will be no shell crossing singularity even if $R' + R\nu' = 0$. Here ρ is finite and $R' + R\nu' = 0$ will correspond to a real extrema for β . On the other hand if $\frac{R'}{R} = \sqrt{\sum B_i'^2 - 4A'C'}$ then shell crossing singularity is a 2-surface in $(n+2)$ -dimensional space-time. For $\frac{R'}{R} < \sqrt{\sum B_i'^2 - 4A'C'}$,

the shell crossing set lie on a n -hypersphere and it intersects with constant (t, r) along the curve $\frac{R'}{R} = -\nu' = \text{constant}$. For future work it will be interesting to study in details the possibility of shell crossing singularity with pressure.

Appendix: A detailed study of a quadratic expression:

Consider a general quadratic expression in n variables

$$z = a \sum_{i=1}^n x_i^2 + \sum_{i=1}^n b_i x_i + c \quad (37)$$

For the present problem (given in equation (31)) we have

$$a = R'A - RA' , \quad b_i = R'B_i - RB'_i , \quad c = CR' - C'R \quad (38)$$

Equation (37) can be rewritten as

$$z = a \sum_{i=1}^n \left(x_i + \frac{b_i}{2a} \right)^2 + \frac{d}{4a} , \quad d = 4ac - \sum_{i=1}^n b_i^2 \quad (39)$$

By substitution from (38) we obtain

$$d = MR^2 + NR'^2 + LRR' \quad (40)$$

with

$$M = 4A'C' - \sum_{i=1}^n B_i'^2 , \quad N = 4AC - \sum_{i=1}^n B_i^2 , \quad L = -4(A'C + AC') + 2 \sum_{i=1}^n B_i B'_i$$

Using equation (6) one sees that $N = 1$ and $L = 0$ and d simplifies to

$$d = MR^2 + R'^2$$

We shall now discuss the sign of z for the following cases:

(i) $a > 0, d > 0$: It is clear from equation (39) that z is positive definite for all values of x_i 's.

(ii) $a > 0, d = 0$: In this case $z \geq 0$. The equality sign occurs for a particular value of the variables x_i 's namely $x_i = -\frac{b_i}{2a}$, $i = 1, 2, \dots, n$.

(iii) $a > 0, d < 0$: Here z has indefinite sign. In particular, z will zero when the variables will lie on a hypersphere having centre $(-\frac{b_1}{2a}, -\frac{b_2}{2a}, \dots, -\frac{b_n}{2a})$ and radius $\frac{\sqrt{|d|}}{2a}$.

If $a < 0$ we have indefinite sign of z for $d < 0$ and for $d > 0$, z will be negative definite (which is not possible in the present paper). We note that if $z = 0$ (with $d < 0$) then as above the variables x_i 's lie on the hypersphere having the same centre and radius is $\frac{\sqrt{|d|}}{2a}$. But if the variables lie inside the above hypersphere then z will have positive value.

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